INPLANE FLEXURAL VIBRATION OF STIFFENED CIRCULAR RINGS

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Certified that this work has been carried out under my supervision and that the work has not been submitted elsewhere for a degree.

(Dr. V. Sundararajan)

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NOMENCLATURE

E	Modulus of elasticity of the material of ring
G_{rs}	Matrix
h	Special value of l where a concentrated load is applied
I	Moment of inertia of an element of the ring
i	Integer
j	Integer - also used as $(-1)^{\frac{1}{2}}$
$\underline{k}(l,m)$	Matrix
k'	Generalised elastic influence function
$k_1(l^{(i)},0)$	Radial deflection of a freely supported ring segment at $l^{(i)}$ due to a unit moment acting at $l = 0$
$k_1(l^{(i)},l_1)$	Radial deflection of a freely supported ring segment at $l^{(i)}$ due to a unit moment acting at $l = l_1$
$k_0(l^{(i)},h)$	Radial deflection of a freely supported ring segment at $l^{(i)}$ due to a unit concentrated radial load acting at $l = h$
L	Span length
l	Dimensionless circumferantial coordinate
l_1	Support spacing - dimensionless
$l^{(i)}$	Local circumferential coordinate
$M^{(i)}$	Bending moment at the $i^{th.}$ support
m	Circumferential coordinate (non-dimensional) giving the position of unit load
N	Integer, denoting number of spans

n	Integer
p_{σ}	Transformed coordinate (a function of σ)
q	Mass per unit of length
R	Radius of the ring
R_L	Reaction at the left hand support
r	Integer
s	Integer
$s_1(0,l_1)$	Radial slope at l_1 i.e. at the right hand support of a freely supported ring segment due to a unit moment acting at the $l = 0$ i.e. at the left hand support
$s_1(0,0)$	Radial slope at $l = 0$ i.e. at the left hand support of a freely supported ring segment due to a unit moment acting at $l = 0$ i.e. at the left hand support itself
$s_0(0,h)$	Radial slope at $l = 0$ i.e. at the left hand support of a freely supported ring segment due to a unit concentrated radial load at $l = h$
$s_0(l,h)$	Radial slope at $l = l_1$ i.e. at the right hand support of a freely supported ring segment due to a unit concentrated radial load acting at $l = h$
t	Time parameter
v	Circumferential coordinate
$v^{(i)}$	Local circumferential coordinate
V	Phase velocity
w	Radial coordinate
$w^{(i)}$	Local radial coordinate
$\underline{w}(l)$	Matrix

$v^{(i)}(l^{(i)},h)$	Radial deflection in the $i^{th.}$ span at any place $l^{(i)}$ due to a unit concentrated radial load on the $0^{th.}$ span at $l = h$
Y	Non - dimensional parameter
σ	Interpanel phase angle
Ω, ω	Frequency
δ	A frequency parameter
δ_{ij}	Kronecker Delta
λ	A dimensionless eigen value parameter

SYNOPSIS

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"INPLANE FLEXURAL VIBRATION OF STIFFENED CIRCULAR RINGS

The present investigation deals with the inplane flexural vibration of a circular ring with large number of equally spaced radial supports. The property of the cyclic structures that all the elements are identical and the last span is adjecent to the first span is made use of. The sole restriction to this method is that the vibration phenomena should be described by linear theory.

The problem of the ring on many supports have been reduced to the analysis of an equivalent single span using the properties of circulant matrix. In this way all the spans vibrate identically but adjecent spans can have a phase difference. Making the radius of the ring infinite, but keeping the span length finite, the case of free vibration of fully infinite beam is arrived at. Galerkin's technique has been used to solve the equation of motion.

Numerical results for the natural frequencies in the case of a stiffened ring with 10 & 20 supports have been calculated for inter - panel phase angles of π, 2π and $\pi/3$.

Natural frequencies in the case of an infinite beam has been obtained for interspan phase angle of π, 2π and $\pi/3$. The results have been compared with the existing values and are found to be in exact agreement.

CHAPTER I

INTRODUCTION

1.1 General:

The problem of vibration of a free - ring is encountered in the analysis of the frequencies of vibration of various kinds of circular frames vis. rotating electric - machines, ball bearings, gears, aircraft structures etc. All these cases are idealised to a circular ring under the assumption that the cross - sectional dimensions of the ring are small in comparison with the radious of its centre line that is to a thin circular ring, and that the cross - section has an axis of symmetry situated in the plane of the ring.

Analysis of rings with radial support (stiffened - rings) finds application in aircraft and missile structures. The bulkhead in the fuselage of an aircraft can be idealised to be a ring on many radial supports. Most of the works done on rings uptil now are on free - rings or on rings on a few number of radial supports. The present work deals mainly with the study of inplane vibration of circular rings on large number of equally spaced radial supports.

1.2 Literature Survey:

Hoppe (1)* investigated the vibration (linear) of thin circular rings in 1871 and was first to get the frequencies and mode shapes. Lord Rayleigh (2) obtained the results by assuming that the mid - surface of the ring had no extension. Later on many investigators (3,4,5) fully analysed the vibration of rings taking into account the effects of the mid - surface extension, rotary inertia and shear deformation. Kuhl (6) in 1942 analysised the case of thick circular rings and Federhofer (7) analysed the non - linear free vibration of rings in 1959. A detailed literature survey on free - rings has been done by Rao (8).

As for the problem of stiffened rings with large number of supports practically no work, to the author's knowledge, appears to have been done so far. However the fundamental work of Brookes Taylor (1713) on the massless elastic string carrying equally spaced point masses (9) which were further extended by Lord Rayleigh (10, 11) in order to generalise the wave propagation phenomena suggested by Raynolds disconnected pendulums, are of great importance in analysing the vibration problems of cyclic and periodic structures like stiffened rings on fairly large number of equally spaced radial supports, fully or semi-infinite continuous beams on many equidistant supports, axially stiffened cylindrical shells etc. Cyclic structures are those

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* Numbers in parentheses refer to the References at the end of the thesis.

which are composed of 'n' identical elements equally spaced around a circle, with the last or n^{th} element adjacent to the first element. Frank Lane (12) made an important contribution by investigating the problem of compressor blades (essentially a cyclic structure) flutter under the assumption that each blade had a finite number of degree of freedom. He was able to show that the flutter analysis of a cyclic structure of this type characterised by a large number of identical fluttering blades could be reduced with no loss of generality whatsoever to the analysis of a "single equivalent blade".

J.W. Miles (13) in his paper on "vibration of beams on many supports" starts with a differential equation of motion and chooses a solution which satisfies the boundary conditions. The solution gives a difference equation which is a recurrence relation between three adjacent supports. The solution of the difference equation gives the result that the frequencies fall in a periodically spaced groups that are separated by spectral gaps of widths equal to approximately half the interval between the natural frequencies of a single beam on a square root frequency scale. These groups tend to uniform spectra as the number of supports tend to infinity, but the gaps remain, giving a band - pass - character to the entire spectrum. Wave propagation along an infinite, periodically supported beam is also discussed and the phase and group velocities are evaluated as functions of frequency.

R.O. Stearman (14) has analysed the flutter problem of stiffened cylindrical shells on many radial supports. Lane's principle that all the system mode shapes of an n-bladed system

can be obtained in terms of n - single "equivalent blades" has been extended to the case of continuous or infinite number of degree of freedom systems (Appendix - 1). Both Lane and Stearman have made use of the recurrent characteristic of the cyclic structures analysed by them and have made use of an important physical condition that the influence of the neighbouring spans becomes less and less with increasing distance. This gives a tremendous simplification for cyclic and periodic configurations with fairly large number of supports.

Stearman takes the Donnel's cylindrical shell equation, makes use of the reduction procedure laid down by Lane and extended by himself to uncouple the equations and finally indicates only the method of solutions without giving any numerical results. He replaces the operation of solving the n - eigen value problems (thus obtained after uncoupling) by a minimization process with respect to an interpanel phase angle. A justification for this minimization process is given in (12). The results thus obtained are valid when all panels of the original problem flutter at the same frequency but in different modes and also different phase shifts between different panels, as well as when all panels flutter in the same mode shape but differ by a constant phase shift between different panels. The basic difference in the approaches of Miles and Stearman is that the former starts with the differential equation of motion where as the later begins with the integral mode of equation.

The only available work up-to-date on the vibration of stiffened rings in particular is of Rao and Sundararajan (8).

But this investigation also goes only upto four supports. Here they have derived the equilibrium equations for the inplane vibration of a ring taking into account the effect of rotatory inertia and shear in the form:

$$\frac{\partial^6 \omega}{\partial \theta^6} + 2\frac{\partial^4 \omega}{\partial \theta^4} + \frac{\partial^2 \omega}{\partial \theta^2} = -\frac{rAR^4}{EIg}\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 \omega}{\partial \theta^2} - \omega\right) \qquad (1.1)$$

A solution of the form

$$w(\theta,t) = W(\theta)\, e^{j\omega t} \qquad (1.2)$$

and

$$W(\theta) = \sum^{6} C_i\, e^{p_i \theta} \qquad (1.3)$$

is assumed where C_i's are some constants to be evaluated by using boundary conditions and p_i's are the six roots of the polynomial

$$p^6 + 2p^4 + k_1 p^2 + k_2 = 0 \qquad (1.4)$$

By satisfying the boundary conditions, the equation gives 6N unknowns where N is the number of supports. By setting the determinant of the coefficient matrix equal to zero, it gives the characteristic equation for the square of the frequency. Rao and Sundararajan have analysed rings upto four supports only and in doing so they end up with a 24 X 24 matrix equation. The order of the matrix equation in this method comes to be equal to 6N X 6N and this suggests the limitations of this method for large N. Natural frequencies have been calculated assuming that the phase difference between the adjacent spans to be equal to π.

1.3 Statement of the Problem:

In the present work we analyse the inplane - flexural vibration of a thin circular ring on fairly large number of equally spaced radial supports. The equation of motion is taken in its integral form and Galerkin's approximate method is used to get the natural frequency of the stiffened ring. The analysis is carried on for the complete solution of a stiffened ring having any phase angle between the two adjecent panels - either π or its integer multiple or otherwise and frequencies are obtained both for stationary as well as travelling mode shapes.

Further, the radius of the ring has been extended to infinity, keeping the span length finite(that is, keeping Rl_1 finite) and this has given the solutions for fully - infinite beam on equally spaced supports. The frequency thus obtained for the infinite beam is compared with the results obtained by Miles. This ascertained the validity of the analysis to considerable extent.

The present analysis is subject to the sole restriction that the phenomenon be satisfactorily described by linear theory and thus the principle of superposition holds. Effects of rotatory inertia and shear deformations have not been taken into account to simplify the analysis. It has also been assumed that the mid - surface of the ring has no circumferential strain.

CHAPTER II

FORMULATION OF THE PROBLEM

The equation of motion in its integral - form for a one dimensional elastic structure is given as

$$w(l) = \Omega^2 \int_0^1 k(l,m)\, q(m)\, w(m)\, dm \tag{2.1}$$

This is an integral equation identified as Fredholm's equation of second kind.

For a single span beam (straight or curved) $w(l)$ is the deflection (vertical or radial) at l, $q(m)$ is the mass per unit of length and $k(l,m)$ is the deflection at l due to a unit concentrated load at m, Ω is the frequency.

In the case of a cyclic stiffened ring the equation of motion for any span can be written in the following way taking into account the effects of all other spans on the particular span under consideration

$$w_1(l) = \Omega^2 \int_0^1 \Big[k_{11}(l,m)\, w_1(m)\, q(m) + k_{12}(l,m)\, w_2(m)\, q(m) + \dots + k_{1n}(l,m)\, w_n(m)\, q(m) \Big] dm \tag{2.2}$$

where $w_1(l)$ = Radial deflection at l of the $1^{th.}$ span.

l = Non - dimensionalised curvilinear coordinate where the deflection is considered.

m = A variable of integration and also the curvilinear coordinate giving the position of the unit force being applied.

$k_{ij}(l,m)$ = Radial deflection of the $i^{th.}$ panel at l due to a unit concentrated load in the $j^{th.}$ panel at m.

Ω = Frequency.

Equation (2.2) can be written as follows in the matrix form

$$\begin{Bmatrix} w_1(l) \\ w_2(l) \\ w_3(l) \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ w_n(l) \end{Bmatrix} = \Omega^2 q(m) \int_0^l \begin{bmatrix} k_{11}(l,m) & k_{12}(l,m) & \cdots & k_{1n}(l,m) \\ k_{21}(l,m) & k_{22}(l,m) & \cdots & k_{2n}(l,m) \\ k_{31}(l,m) & k_{32}(l,m) & \cdots & k_{3n}(l,m) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ k_{n1}(l,m) & k_{n2}(l,m) & \cdots & k_{nn}(l,m) \end{bmatrix} \begin{Bmatrix} w_1(m) \\ w_2(m) \\ w_3(m) \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ w_n(m) \end{Bmatrix} dm \qquad - (2.3)$$

and in the notational form this can be written as

$$\underline{w}(l) = \Omega^2 q(m) \int_0^1 \underline{k}(l,m)\ \underline{w}(m)\ dm \qquad (2.4)$$

where $\underline{k}$ is a square matrix of n^2 elements

& $\underline{v}$ is a column matrix of n elements.

The integration extends over the entire span length i.e. m goes from 0 to 1.

Here we recognise that a stiffened - ring of equal spans is a cyclic structure. Cyclic structures are those which are composed of 'n' identical elements equally spaced around a circle, with the last or $n^{th.}$ element being adjecent to the first element. As such the matrix $\underline{k}$ turns out to be a circulant matrix and thus it has only n unknown elements.

The matrix $\underline{k}$ being a circulant matrix can be reduced to a diagonal form, say $\underline{k}'$, by a collineatory transformation (Appendix - I). Since $\underline{k}$ and $\underline{k}'$ are related by a collineatory transformation, the eigen values of $\underline{k}$ and $\underline{k}'$ are identical. But $\underline{k}'$ is a diagonal matrix, hence the eigen values are given by the n equations.

$$|k'_p| = 0 \quad (p=0,1,2,\ldots\ldots,n-1) \qquad (2.5)$$

Hence the coupled integral problem represented by equation (2.4) can be transformed and written in the form:

$$B(\sigma_i)*p_i = 0 \;;\; \sigma_i = 2\pi i/n \;;\; (i = 0,1,2,\ldots\ldots,n-1) \qquad - (2.6)$$

where

$$B(\sigma_i) = H_1 + H_2 e^{j\sigma_i} + H_n e^{-j\sigma_i} + H_3 e^{j2\sigma_i} + H_{n-1} e^{-j2\sigma_i} + \ldots\ldots \qquad - (2.7)$$

(Ref. Appendix - I)

This shows that the vibration solution of the $i^{th.}$ equivalent span actually corresponds to the solution in which the original vibration problem is solved for one span when this span is under a very special form of influence from all other spans; namely all other spans oscillate with the same span mode and with the same phase shift angle σ_i (yet undetermined) between adjacent panels. The operation of solving the n eigen value problems (2.6) can be replaced by a minimization process with respect to an interpanel phase angle σ as suggested in (12). We remark here that the lowest frequency occurs when σ is equal to π (13 & 14).

The eigen value problem (2.4) after dropping the suffix i and with the time factor removed reduces to the general form:

$$B(\sigma)x\, p_\sigma = p_\sigma(l) - \Omega^2 \int k'(l,m)\, p_\sigma(m)dm = 0 \qquad (2.8)$$

where

$$k' = k_1' + k_2'e^{j\sigma} + k_n'e^{-j\sigma} + k_3'e^{j2\sigma} + k_{n-1}'e^{-j2\sigma} + \dots\dots \qquad (2.9)$$

where

$k_i'(l,m)$ represents the influence functions giving the deflection curve of the $(s + 1 - i)^{th.}$ span $(s \geq i)$ or $(s + 1 - i + n)^{th.}$ span $(s < i)$ when a concentrated load is applied to the $s^{th.}$ panel.

Now once the influence function (Green's function) of the stiffened - ring is determined k_i' will be known.

We now renumber the panels as shown in Fig. (2). This will help us in determining the function k'. This function k' may be thought of as a generalised elastic influence function yielding the lateral deflection curve of the equivalent span (denoted by the number zero) when the panel is under a very special form of influence from all other panels. The concentrated loads in this case are radial and occupy the same position relative to each span (Fig. 2).

Hence it only remains to find the generalised influence function k' and then solving the integral equation (2.8).

Without any loss of generality we shall rename the transformed coordinate system as v and w (i.e. w being used for p) and shall work within this frame of transformed coordinate system henceforth.

CHAPTER III

SOLUTION OF THE PROBLEM

The solution of the transformed integral equation (2.8) at first calls for the determination of the generalised Green's function (k') for a stiffened ring and then the solution of the integral equation itself to get the frequency.

A. Determination of the Green's Function for a Cyclic Stiffened Ring:

The Green's function yields the radial displacement component at any point on the stiffened ring due to a concentrated radial load at any point on any span. We assume that the ring is simply supported on equally spaced radial supports. These supports are assumed to prevent radial and tangential deflection but offer no rotational constraint to the ring segment. We remark here that our analysis is valid only in the domain of linearity hence principle of superposition holds.

Let the cyclic stiffened ring be of radius R and each span length equal to Rl_1. When one of the spans is acted upon by a concentrated load, the deflections of this span induce deflections in its neighbouring spans which will in turn influence their neighbour, etc. But the radial supports, however tend to damp out this influence and at a sufficient number of spans away from the loaded span no appreciable

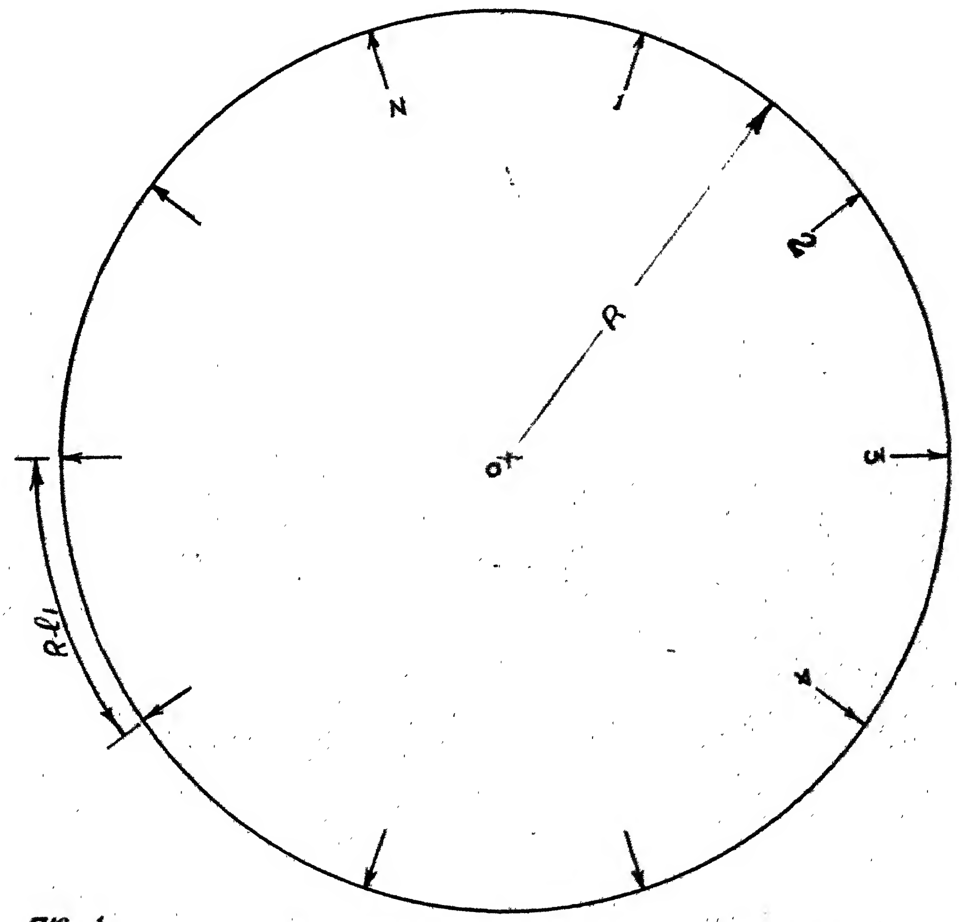

FIG. 1, RING WITH N EQUALLY SPACED RADIAL SUPPORTS

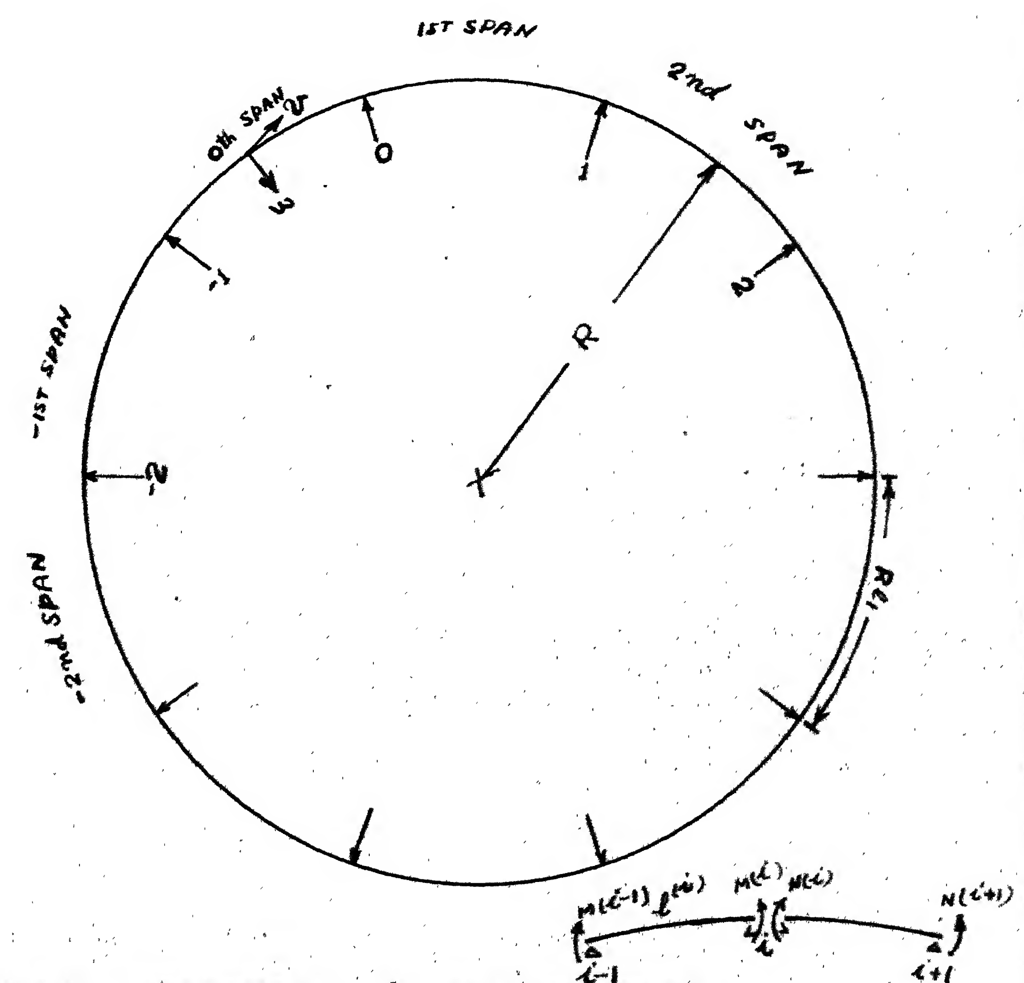

FIG. 2, NOTATIONS & COORDINATES

deflection will be observed. This physical requirement is important and gives one of the conditions for solving the recurrence relation to be encountered later.

The bending of each span of such a loaded ring can be investigated by superposing a solution for freely supported ring segment under a concentrated load and a solution for a similar ring segment bent by unit moment at the support edge. Thus the Green's function for the complete ring can be obtained from these two fundamental solutions as the principal of superposition holds.

In our further analysis l is the non-dimensionalised circumferential coordinate. The $i^{th.}$ span of the ring is described by l in the interval $l^{(i)} \leq l \leq l^{(i+1)}$ or in other words $l^{(i)}$ is a local coordinate system.

We define our displacement coordinate system in the following way

$v^{(i)}$ - Tangential deflection

$w^{(i)}$ - Radial deflection

The superscript (i) indicates that the point under consideration is in the $i^{th.}$ span.

The complete set of the boundary conditions for a stiffened ring is given as follows (Fig.1)

$$\begin{aligned} w^{(i)}(0) &= 0 \quad - (a) \\ v^{(i)}(0) &= 0 \quad - (b) \\ w^{(i)}(l_1) &= 0 \quad - (c) \\ v^{(i)}(l_1) &= 0 \quad - (d) \end{aligned} \qquad (3.1)$$

The matching conditions at the $i^{th.}$ span are given as (Fig. 1)

$$w^{(i+1)}(0) = w^{(i)}(l_1) = 0 \qquad (3.2)$$

$$\frac{dw^{(i+1)}}{dl^{(i+1)}}(0) = \frac{dw^{(i)}}{dl^{(i)}}(l_1) \qquad (3.3)$$

$$\frac{d^2w^{(i+1)}}{(dl^{(i+1)})^2}(0) = \frac{d^2w^{(i)}}{(dl^{(i)})^2}(l_1) \qquad (3.4)$$

To determine the Green's function it will be sufficient to find the radial deflection in the $i^{th.}$ span due to a unit concentrated radial load acting on the $0^{th.}$ span at $l = h$, and the deflection can be given as

$$w^{(i)}(l^{(i)},h) = M^{(i-1)}\, k_1(l^{(i)},0) - M^{(i)}\, k_1(l^{(i)},l_1) + \delta_{i0}\, k_0(l^{(i)},h) \qquad (3.5)$$

where

$w^{(i)}(l^{(i)},h)$ = Radial deflection in the $i^{th.}$ span at any place $l^{(i)}$ (the superscript i is given to indicate that the non-diamensionalised parameter is taking its values in the $i^{th.}$ span) due to a unit concentrated radial load on the $0^{th.}$ span at $l = h$.

$M^{(i-1)}$ = Moment on the $(i-1)^{th.}$ support.

$M^{(i)}$ = Moment on the $i^{th.}$ support.

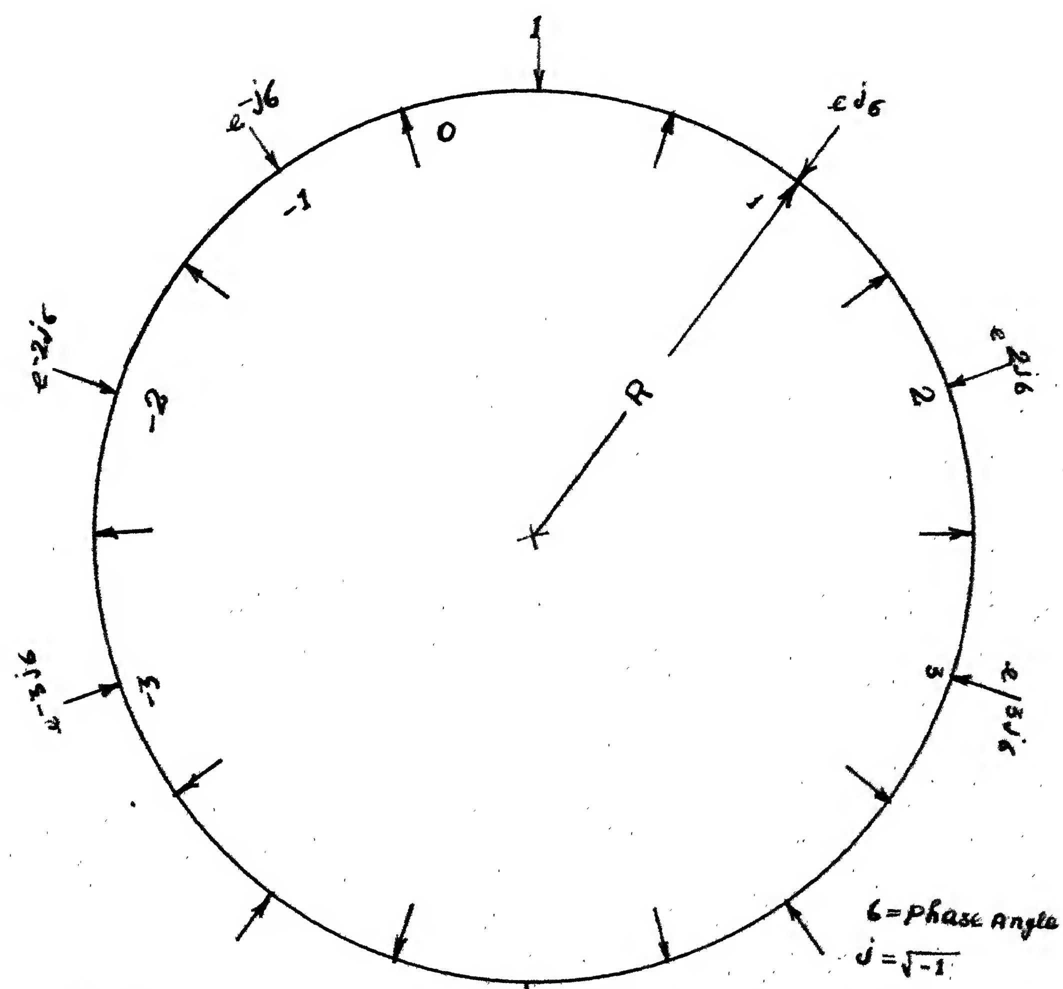

FIG. 3 EQUIVALENT LOADING FOR REDUCED PROBLEM

$k_1(l^{(i)},0)$ = Radial deflection of a freely supported ring segment at $l^{(i)}$ due to a unit moment acting at $l = 0$.

$k_1(l^{(i)},l_1)$ = Radial deflection of a freely supported ring segment at $l^{(i)}$ due to a unit moment acting at $l = l_1$.

$k_0(l^{(i)},h)$ = Radial deflection of a freely supported ring segment at $l^{(i)}$ due to a unit concentrated radial load acting at $l = h$.

δ_{ij} = Kronecker Delta

that is $\delta_{i0} = 1$ when $i = 0$

$= 0$ when $i \neq 0$

In equation (3.5) the boundary conditions (3.1a and 3.1c) and the matching condition (3.2) are automatically satisfied as k_1 and k_0 satisfy them inherently. The final matching conditions (3.3) of the continuity of slope at the supports are satisfied by following equation if $i \neq 0$ or -1 (Fig. 2).

$$M^{(i-1)}\, s_1(0,l_1) - 2M^{(i)}\, s_1(0,0) + M^{(i+1)}\, s_1(0,l_1) = 0 \qquad (3.6)$$

where

$s_1(0,l_1)$ = Radial slope at l_1, that is at the right hand support of a freely supported ring segment due to a unit moment acting at the left hand support, that is $l = 0$.

$$= \frac{dk_1}{dl}(0,l_1)$$

$s_1(0,0)$ = Radial slope at $l = 0$, that is at the left hand support of a freely supported ring segment due to a unit moment acting at $l = 0$, that is at the left hand support itself.

Similarly the continuity of slope across the $0^{th.}$ and $-1^{st.}$ supports gives the following equations respectively:

$$M^{(1)} s_1(0,l_1) - 2M^{(-1)} s_1(0,0) + M^{(0)} s_1(0,l_1) - s_0(l_1,h) = 0 \qquad - (3.7)$$

and

$$M^{(-2)} s_1(0,l_1) - 2M^{(-1)} s_1(0,0) + M^{(0)} s_1(0,l_1) - s_0(0,h) = 0 \qquad - (3.8)$$

where

$s_0(l_1,h)$ = Radial slope at $l = l_1$, that is at the right hand support of a freely supported ring segment due to a unit concentrated radial load acting at $l = h$.

$$= \frac{dk_0}{dl}(l_1)$$

$s_0(0,h)$ = Radial slope at $l = 0$, that is at the left hand support of a freely supported ring segment due to a unit concentrated radial load at $l = h$.

$$= \frac{dk_0}{dl}(0)$$

Equation (3.6) is a recurrence relation between any three adjacent support moments of the stiffened ring (excepting for

$i = 0$ and -1) and it is solved under the following two side conditions

(1) $|M^{(i+1)}| < |M^{(i)}|$ for i positive, that is the magnitude of the support moment decreases as i increases. Or as a corollary to this if $i \rightarrow \infty$ $M^{(i)} \rightarrow 0$.

(2) Moments at the right hand supports will be considered as a function of only $M^{(0)}$ and those at left hand supports as a function of $M^{(-1)}$.

Under the above two side conditions which are from the physical requirements of stiffened rings, the system of equations at (3.6, 3.7, and 3.8) are solved to get the support moments.

Equation (3.6) is solved as follows for $i \geq 0$ and for $i \leq -1$ separately.

$\underline{i \geq 0}$

Let $M^{(i)} = A\, r^{i}$ (3.9)

then equation (3.6) can be written as

$$A\, r^{i-1}\, s_1(0,l_1) - 2A\, r^{i}\, s_1(0,0) + A\, r^{i+1}\, s_1(0,l_1) = 0$$

or $$s_1(0,l_1)\, r^2 - 2\, s_1(0,0)\, r + s_1(0,l_1) = 0$$

or

$$r = \frac{s_1(0,0)}{s_1(0,l_1)} \pm \sqrt{\left[\frac{s_1(0,0)}{s_1(0,l_1)}\right]^2 - 1} \qquad (3.10)$$

Hence

$$M^{(i)} = A_1\left\{\frac{s_1(0,0)}{s_1(0,l_1)} + \sqrt{\left[\frac{s_1(0,0)}{s_1(0,l_1)}\right]^2 - 1}\right\}^i \qquad (3.11)$$

$$+ A_2\left\{\frac{s_1(0,0)}{s_1(0,l_1)} - \sqrt{\left[\frac{s_1(0,0)}{s_1(0,l_1)}\right]^2 - 1}\right\}^i$$

for $i \geqslant 0$

We know that $\frac{s_1(0,0)}{s_1(0,l_1)} < -1$ and also $M^{(i)} \rightarrow 0$ as $i \rightarrow \infty$.

Hence $A_2 = 0$. Therefore

$$M^{(i)} = A_1\left\{\frac{s_1(0,0)}{s_1(0,l_1)} + \sqrt{\left[\frac{s_1(0,0)}{s_1(0,l_1)}\right]^2 - 1}\right\}^i \qquad (3.12)$$

Equation (3.12) gives that $M^{(0)} = A_1$

Hence

$$M^{(i)} = M^{(0)}\left\{\frac{s_1(0,0)}{s_1(0,l_1)} + \sqrt{\left[\frac{s_1(0,0)}{s_1(0,l_1)}\right]^2 - 1}\right\}^i \qquad (3.13)$$

<u>$i \leqslant -1$</u>

Putting $M^{(-i)} = B\, r^{(i-1)}$ and proceeding in the same fashion as above we get

$$M^{(-i)} = M^{(-1)}\left\{\frac{s_1(0,0)}{s_1(0,l_1)} + \sqrt{\left[\frac{s_1(0,0)}{s_1(0,l_1)}\right]^2 - 1}\right\}^{i-1} \qquad (3.14)$$

This leaves us with four equations namely 3.7, 3.8, 3.13 and 3.14 and from these equations $M^{(0)}$ and $M^{(-1)}$ are

determined which are still unknown.

Let us denote $\dfrac{s_1(0,0)}{s_1(0,l_1)} + \sqrt{\left[\dfrac{s_1(0,0)}{s_1(0,l_1)}\right]^2 - 1}$ by Y to shorten the writing.

Putting the values of $M^{(1)}$ from equation (3.13) and of $M^{(-2)}$ from equation (3.14) in equations 3.7 and 3.8 gives

$$s_1(0,l_1)\, M^{(0)}\, Y - 2\, M^{(0)}\, s_1(0,0) + M^{(-1)}\, s_1(0,l_1) = -\, s_0(l_1,h) \qquad - (3.15)$$

and

$$s_1(0,l_1)\, M^{(-1)}\, Y - 2\, M^{(-1)}\, s_1(0,0) + M^{(0)}\, s_1(0,l_1) = s_0(0,h) \qquad - (3.16)$$

Solving (3.15) and (3.16) simulteneously gives

$$M^{(0)} = -\frac{s_0(l_1,h)\left\{s_1(0,l_1)\, Y - 2\, s_1(0,0)\right\} + s_0(0,h) \times s_1(0,l_1)}{\left\{s_1(0,l_1)\, Y - 2\, s_1(0,0)\right\}^2 - \left\{s_1(0,l_1)\right\}^2} \qquad - (3.17)$$

and

$$M^{(-1)} = \frac{s_0(0,h)\left\{s_1(0,l_1)\, Y - 2\, s_1(0,0)\right\} + s_0(l_1,h) \times s_1(0,l_1)}{\left\{s_1(0,l_1)\, Y - 2\, s_1(0,0)\right\}^2 - \left\{s_1(0,l_1)\right\}^2} \qquad - (3.18)$$

This completely determines all the support moments of a stiffened ring due to a unit concentrated radial load on the $0^{th.}$ span. These equations shall continue to hold even when $R \to \infty$, such that Rl_1 remains finite i.e. the equations hold good for fully - infinite continuous beams of equal spans.

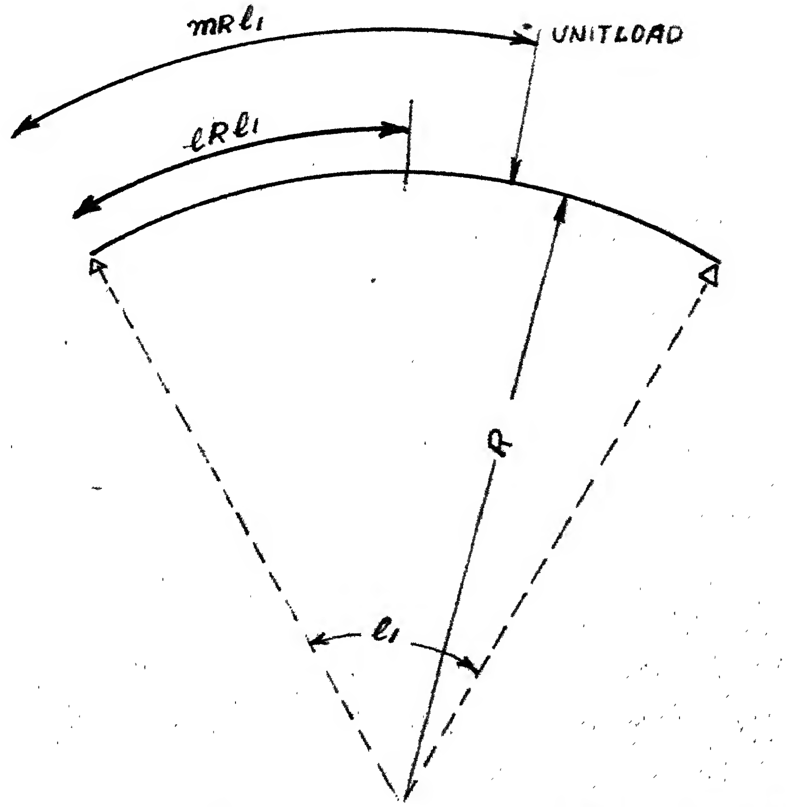

FIG 4, INFLUENCE FUNCTION FOR A CURVED BEAM

B. Determination of Influence Functions for a Simply Supported Curved Beam (Circular Arc)

Equations for k_0 and s_0

The flexural equation for curved beam (circular) is given as

$$\frac{E\,I}{(R\,l_1)^2}\left(\frac{d^2w}{dl^2} + w\,l_1^2\right) = -M \tag{3.19}$$

where E = Modulus of elasticity.

I = Moment of inertia.

M = Moment at a particular cross - section.

The bending moment equation for $0 \leq l \leq m$ and $m \leq l \leq 1$ are given as (Fig. 4) as follows:

$$\frac{E\,I}{(R\,l_1)^2}\left(\frac{d^2w}{dl^2} + w\,l_1^2\right) = -\; R_L\,R\,\mathrm{Sin}(l_1 l) \tag{3.20}$$

and

$$\frac{E\,I}{(R\,l_1)^2}\left(\frac{d^2w}{dl^2} + w\,l_1^2\right) = -\left[R_L\,R\,\mathrm{Sin}(l_1 l) - R\,\mathrm{Sin}\,l_1(l - m)\right] \tag{3.21}$$

respectively. (m) is the position coordinate for the unit concentrated radial load on the beam.

R_L = Reaction at the left hand radial support

$$= \frac{\mathrm{Sin}\,l_1(1 - m)}{\mathrm{Sin}(l_1)} \tag{3.22}$$

Solution of equation (3.20) gives

$$w = R\frac{(R\,l_1)^2}{E\,I}\left[(A_1\,\mathrm{Cos}(l_1 l) + A_2\,\mathrm{Sin}(l_1 l) - \frac{C_1\,l}{2\,l_1}\,\mathrm{Cos}(l_1 l)\right] \quad - (3.23)$$

where

$$C_1 = -\frac{\mathrm{Sin}\,l_1(1-m)}{\mathrm{Sin}(l_1)} \quad (3.24)$$

and equation (3.21) gives

$$w = R\frac{(R\,l_1)^2}{E\,I}\left[A_3\,\mathrm{Cos}(l_1 l) + A_4\,\mathrm{Sin}(l_1 l) - \frac{C_1\,l}{2\,l_1}\,\mathrm{Cos}(l_1 l) - \frac{1}{2\,l_1}\,\mathrm{Cos}\,l_1(1-m)\right] \quad - (3.25)$$

The constants A_1, A_2, A_3 and A_4 are evaluated from the following boundary conditions:

(i) $w = 0$ at $l = 0$

(ii) $w = 0$ at $l = 1$

(iii) w at m is the same from equation (3.20) and (3.21)

and (iv) Radial slope is the same from equation (3.20) and (3.21) at m.

This gives

$$A_1 = 0 \quad (3.26)$$

$$A_2 = \left[A_3\,\mathrm{Cos}(l_1 m) + A_4\,\mathrm{Sin}(l_1 m) - \frac{m}{2\,l_1}\right]\frac{1}{\mathrm{Sin}(l_1 m)} \quad (3.27)$$

$$A_3 = \frac{1}{2\,l_1}\left[m\,\mathrm{Cos}(l_1 m) - \frac{\mathrm{Sin}(l_1 m)}{l_1}\right] \quad (3.28)$$

$$A_4 = \left[\frac{C_1}{2\,l_1}\,\mathrm{Cos}(l_1) + \frac{1}{2\,l_1}\,\mathrm{Cos}\,(1-m)l_1 - A_3\,\mathrm{Cos}(l_1)\right]\frac{1}{\mathrm{Sin}(l_1)}$$

$$- (3.29)$$

Therefore

$$k_0(l^{(i)},h) = \frac{(R\,l_1)^3}{E\,I}\left[\,A_2\,\frac{\mathrm{Sin}(l_1 l^{(i)})}{l_1} - \frac{C_1\,l^{(i)}}{2\,l_1^2}\,\mathrm{Cos}(l_1 l^{(i)})\,\right]$$

$$- (3.30)$$

for $0 \leq l^{(i)} \leq h$

$$= \frac{(R\,l_1)^3}{E\,I}\left[\left\{A_3\,\mathrm{Cos}(l_1 l^{(i)}) + A_4\,\mathrm{Sin}(l_1 l^{(i)})\right\}\frac{1}{l_1} - \frac{C_1\,l^{(i)}}{2\,l_1^2}\,\mathrm{Cos}(l_1 l^{(i)}) - \frac{l^{(i)}}{2\,l_1^2}\,\mathrm{Cos}\,l_1(l^{(i)}-h)\,\right]$$

$$- (3.31)$$

for $h \leq l^{(i)} \leq 1$

and

$$s_0(0,h) = \frac{(R\,l_1)^2}{E\,I}\left[\,A_2 - \frac{C_1}{2\,l_1^2}\,\right] \qquad (3.32)$$

$$s_0\,(l_1,h) = \frac{(R\,l_1)^2}{E\,I}\left[-A_3\,\mathrm{Sin}(l_1) + A_4\,\mathrm{Cos}(l_1) - \frac{C_1}{(2\,l_1^2)}\mathrm{Cos}(l_1) + \frac{C_1}{2\,l_1}\,\mathrm{Sin}(l_1) - \frac{1}{2\,l_1^2}\,\mathrm{Cos}\,l_1(1-h) + \frac{1}{2\,l_1}\,\mathrm{Sin}\,l_1(1-h)\right]$$

$$- (3.33)$$

$$A_4 = \left[\frac{C_1}{2\,l_1}\,\mathrm{Cos}(l_1) + \frac{1}{2\,l_1}\,\mathrm{Cos}\,(1-h)l_1 - A_3\,\mathrm{Cos}(l_1)\right]\frac{1}{\mathrm{Sin}(l_1)}$$

$$- \quad (3.29)$$

Therefore

$$k_0(l^{(i)},h) = \frac{(R\,l_1)^3}{E\,I}\left[\,A_2\,\frac{\mathrm{Sin}(l_1 l^{(i)})}{l_1} - \frac{C_1\,l^{(i)}}{2\,l_1^2}\,\mathrm{Cos}(l_1 l^{(i)})\,\right]$$

$$- \quad (3.30)$$

$$\text{for} \quad 0 \le l^{(i)} \le h$$

$$= \frac{(R\,l_1)^3}{E\,I}\left[\left\{A_3\,\mathrm{Cos}(l_1 l^{(i)}) + A_4\,\mathrm{Sin}(l_1 l^{(i)})\right\}\frac{1}{l_1}\right.$$

$$\left. - \frac{C_1\,l^{(i)}}{2\,l_1^2}\,\mathrm{Cos}(l_1 l^{(i)}) - \frac{l^{(i)}}{2\,l_1^2}\,\mathrm{Cos}\,l_1(l^{(i)}-h)\,\right]$$

$$- \quad (3.31)$$

$$\text{for} \quad h \le l^{(i)} \le 1$$

and

$$s_0(0,h) = \frac{(R\,l_1)^2}{E\,I}\left[\,A_2 - \frac{C_1}{2\,l_1^2}\,\right] \qquad (3.32)$$

$$s_0\,(l_1,h) = \frac{(R\,l_1)^2}{E\,I}\left[\,-A_3\,\mathrm{Sin}(l_1) + A_4\,\mathrm{Cos}(l_1) - \frac{C_1}{(2\,l_1^2)}\mathrm{Cos}(l_1)\right.$$

$$\left. + \frac{C_1}{2\,l_1}\,\mathrm{Sin}(l_1) - \frac{1}{2\,l_1^2}\,\mathrm{Cos}\,l_1(1-h) + \frac{1}{2\,l_1}\,\mathrm{Sin}\,l_1(1-h)\right]$$

$$- \quad (3.33)$$

Equations for k_1 and s_1

Similarly starting with the bending moment equation (3.19), the influence functions for a simply supported ring segment acted upon by a unit moment at the supports are determined as follows:

$$k_1(l^{(1)},0) = \frac{(R\,l_1)^2}{E\,I}\Bigg[\frac{\mathrm{Cos}(l_1 l^{(1)})}{l_1^2}\left\{1 - \frac{l_1 l^{(1)}}{2\,\mathrm{Sin}(l_1)}\right\}$$

$$+ \frac{\mathrm{Sin}(l_1 l^{(1)})}{l_1^2}\left\{\frac{1}{\mathrm{Sin}(l_1)} + \frac{l_1\,\mathrm{Cos}(l_1)}{2\,\mathrm{Sin}^2(l_1)} - \frac{\mathrm{Cos}(l_1)}{\mathrm{Sin}(l_1)}\right\} - \frac{1}{l_1^2}\Bigg] \quad - (3.34)$$

$$s_1(0,l_1) = \frac{R\,l_1}{E\,I}\left[\frac{1}{2} - \frac{\mathrm{Sin}(l_1)}{l_1} + \frac{\mathrm{Cos}(l_1)}{2\,l_1\,\mathrm{Sin}(l_1)} + \frac{\mathrm{Cos}^2(l_1)}{2\,\mathrm{Sin}^2(l_1)} - \frac{\mathrm{Cos}^2(l_1)}{l_1\,\mathrm{Sin}(l_1)}\right] \quad - (3.35)$$

$$s_1(0,0) = \frac{R\,l_1}{E\,I}\left[\frac{1}{2\,l_1\,\mathrm{Sin}(l_1)} + \frac{\mathrm{Cos}(l_1)}{2\,\mathrm{Sin}^2(l_1)} - \frac{\mathrm{Cos}(l_1)}{l_1\,\mathrm{Sin}(l_1)}\right] \quad (3.36)$$

The trigonometric functions in equations (3.30 to 3.36) are expanded in terms of the angles and setting $l_1 = 0$ (i.e. $R = \infty$, such that Rl_1 is finite) the following equations for straight beam are obtained thus also confirming the correctness of equation (3.30 to 3.36)

$$k_0(l^{(1)},h) = \frac{(R\,l_1)^3}{E\,I}\left[l^{(1)}\,\frac{(1-h)}{6}\left\{2h - h^2 - (l^{(1)})^2\right\}\right] \quad (3.37)$$

for $0 \le l^{(1)} \le h$

$$= \frac{(R\,l_1)^3}{E\,I}\left[\frac{1}{6}\left\{l^{(i)}\left((1-h)-(1-h)^3\right)\right.\right.$$

$$\left.\left. -\,(1-h)\,(l^{(i)})^3 + (l^{(i)}-h)^3\right\}\right] \qquad (3.38)$$

for $h \leq l^{(i)} \leq 1$

$$s_0(0,h) = \frac{(R\,l_1)^2}{E\,I}\cdot\frac{1}{6}\cdot h\,(1-h)\,(2-h) \qquad (3.39)$$

$$s_0(l_1,h) = -\,\frac{(R\,l_1)^2}{E\,I}\cdot\frac{1}{6}\cdot h\,(1-h)\,(1+h) \qquad (3.40)$$

$$k_1(l^{(i)},0) = \frac{(R\,l_1)^2}{E\,I}\cdot\frac{l^{(i)}}{6}\quad(1-l^{(i)})\,(2-l^{(i)}) \qquad (3.41)$$

$$s_1(0,l_1) = -\,\frac{R\,l_1}{E\,I}\;\frac{1}{6} \qquad (3.42)$$

$$s_1(0,0) = \frac{R\,l_1}{E\,I}\;\frac{1}{3} \qquad (3.43)$$

Having determined all the influence functions and the support moments, the radial deflection - curve of any $i^{th.}$ span (Ref. equation 3.5) can be obtained completely from the following equations for the right hand spans and left hand spans respectively.

For $i \geqslant 0$

$$w^{(i)}(l^{(i)},h) = M^{(0)}\,Y^{i-1}\left\{k_1(l^{(i)},0) - Y\,k_1(l^{(i)},l_1)\right\}$$

$$+\,\delta_{i0}\,k_0(l^{(i)},h) \qquad (3.44)$$

and for $i \leq -1$

$$w^{(i)}(l^{(i)},h) = M^{(-1)}\, Y^{i-1}\left\{ Y\, k_1(l^{(i)},0) - k_1(l^{(i)},l_1)\right\} + \delta_{i0}\, k_0(l^{(i)},h) \qquad (3.45)$$

Referring back to equation (2.7) and (Fig. 3) giving the special loading obtained after transformation, the generalised influence function k' can be obtained as follows by superimposing the effects of the concentrated loads and employing the cyclic property of the stiffened ring, having a total of $2N + 1$ spans

$$\begin{aligned} k' = M^{(-1)} &\left[k_1(l^{(i)},0) + \left\{ Y\, k_1(l^{(i)},0) - k_1(l^{(i)},l_1)\right\} \times \sum_{i=1}^{N} Y^{i-1}\, e^{j(i\sigma)}\right] + M^{(0)}\left[- k_1(l^{(i)},l_1) + \left\{ k_1(l^{(i)},0) \right.\right. \\ &\left.\left. - Y\, k_1(l^{(i)},l_1)\right\} \times \sum_{i=1}^{N} Y^{i-1}\, e^{-j(i\sigma)}\right] + k_0(l^{(i)},h) \end{aligned} \qquad - (3.46)$$

This completely determines the influence function for a stiffened ring.

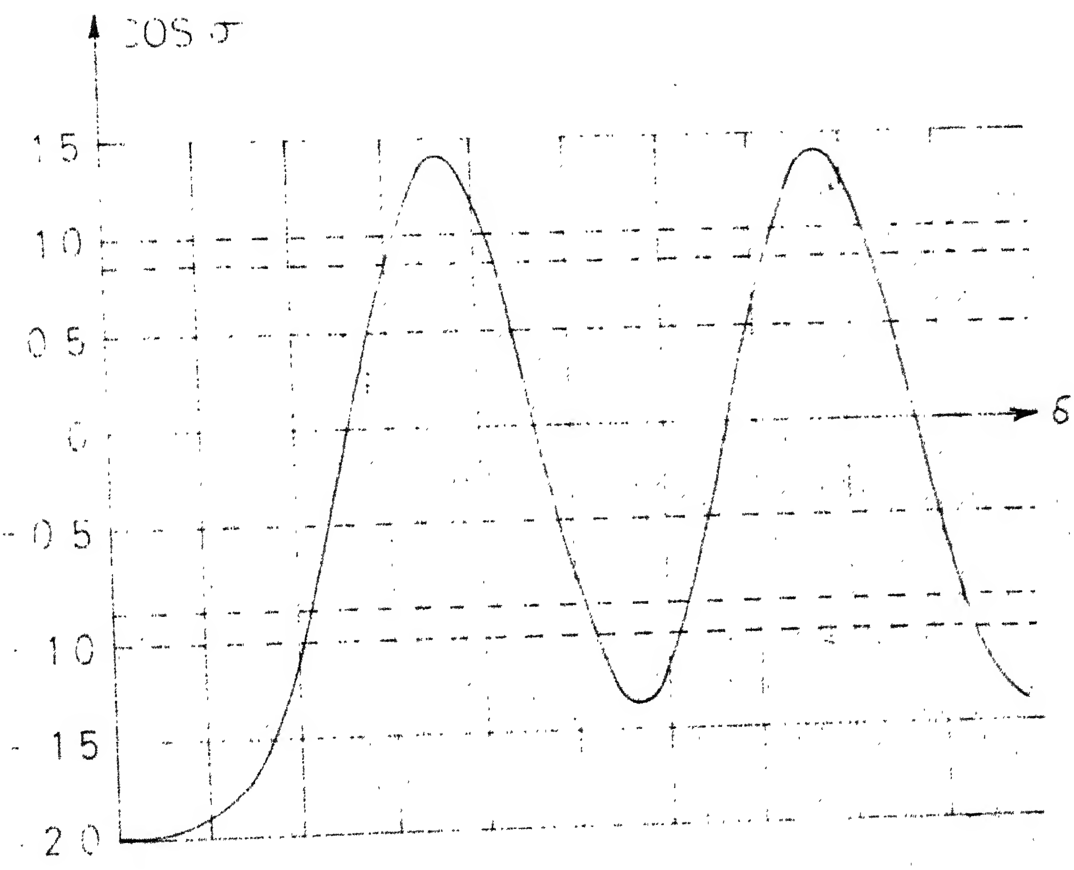

FIG. 5 ILLUSTRATING THE PERMISSIBLE VALUES OF δ AND σ

C. Solution of the Integral Equation:

Galerkin's method is used to solve the integral equation (2.8) and to obtain the minimum frequency. The assumed eigen function or the deflection shape is taken as the real part of equation (A2 - 5) (see - Appendix II). Miles (13) has derived these functions such that they satisfy all the boundary conditions also from equation (3.1) to equation (3.4). The eigen function assumed is

$$w_{\sigma}(l) = \sum_{s=1}^{N} A_s \left[f_s(l) + \mathrm{Cos}\ \sigma\, f_s(1 - l) \right] \tag{3.47}$$

and it is continuous, integrable and continuously differentiable in the area of interest. This gives the inextensional vibration mode shapes of a stiffened ring.

where

$$f_s(l) = \mathrm{Sinh}\ \delta(s)\ \times \mathrm{Sin}\ \delta(s)l - \mathrm{Sin}\ \delta(s) \times \mathrm{Sinh}\ \delta(s)l \tag{3.48}$$

and
$$\mathrm{Cos}\,\sigma = \frac{\mathrm{Sinh}\,\delta(s)\ \mathrm{Cos}\,\delta(s) - \mathrm{Sin}\,\delta(s)\ \mathrm{Cosh}\,\delta(s)}{\mathrm{Sinh}\,\delta(s) - \mathrm{Sin}\,\delta(s)} \tag{3.49}$$

Equation (3.49) is plotted in (Fig. 5) and the values of $\delta(s)$ is obtained either from the equation (3.49) numerically or from (Fig. 5) for any particular value of σ.

If $\delta(s)$ is an integer multiple of π then the assumed mode shape is taken as

$$w_{\sigma}(l) = \sum_{s=1}^{N} A_s \left[\mathrm{Sin}\ \delta(s)\ l \right] \tag{3.50}$$

where $\cos\sigma = (-1)^s$

and $\delta(s) = s\pi$

The frequency parameter $\delta(s)$, appearing in the span-wise modes, will take all positive values in the above defined interval as the phase angle σ varies continuously from 0 to 2π. There is no restriction of σ in taking any value (as is the case for finite number of supports) as $i \to \pm\infty$ in our case.

Putting the assumed mode shape from equation (3.47) into the integral equation of motion (2.8) and applying the Galerkin's method, the following matrix equation is arrived at

$$[M]\{A\} = \Omega^2 [G]\{A\} \tag{3.51}$$

where

$$M_{rs} = \int_0^1 q(l)\, T_r(l)\, T_s(l)\, dl \tag{3.52}$$

$q(l)$ = mass per unit of length

$$T_r(l) = f_r(l) + \cos\sigma\, f_r(1-l) \tag{3.53}$$

and

$$G_{rs} = \int_0^1 q(l)\, T_r(l) \left[\int_0^1 k'(l,m)\, q(m)\, T_s(m)\, dm\right] dl \tag{3.54}$$

where $k'(l,m)$ is given by equation (3.46)

Equation (3.51) is a classical eigen value problem and can be solved. This completely solves the vibration problem of a stiffened cyclic ring on large number of supports.

CHAPTER IV

NUMERICAL RESULTS AND CONCLUSIONS

The solutions obtained in the previous chapters are most general and give the complete solutions for a stiffened ring with any number of supports above 10. While deriving the equations for support moments (Equation 3.12 and 3.14) the condition that the moments at the right hand supports are a function of only $M^{(0)}$ and those at the left hand supports as a function of only $M^{(-1)}$ was made use of. Thus the equations for the right hand support moments and the left hand support moments were obtained as:

$$M^{(i)} = M^{(0)} Y^{i} \text{ for } i \geqslant 0 \tag{4.1}$$

and

$$M^{(i)} = M^{(-1)} Y^{i-1} \text{ for } i \leqslant -1 \tag{4.2}$$

respectively.

For a straight beam with simply supported ends

$\dfrac{s_1(0,0)}{s_1(0,l_1)}$ is equal to -2.00 and thus Y becomes equal to -0.268.

For a curved beam with circular arc the value of $\dfrac{s_1(0,0)}{s_1(0,l_1)}$

decreases further than that for a straight beam and its value

goes down upto -2.33 for a quarter circular arc. So the value of Y goes as low as -0.225. An inspection of Equation 4.1 and 4.2, keeping in view the values of Y gives the lower limit on the number of supports for a reasonable validity of the solutions obtained in the Chapter III.

The present investigation gives the natural frequency for any interpanel phase angle. It is interesting to note (Appendix - II) that a phase angle of π or its integer multiple gives a stationary mode of vibration where as a phase angle of other than π or its integer multiple represents a travelling mode of vibration.

Numerical test results are obtained for fully infinite continuous beam by making $R \to \infty$ keeping Rl_1 finite. The lowest natural frequencies of the infinite beam for an interpanel phase angle of π, 2π and $\pi/3$ have been obtained and the results are found in exact agreement with the results given by Miles (13)

TABLE 1

FREQUENCIES OF VIBRATION FOR A FULLY INFINITE CONTINUOUS BEAM

Serial Number	Values of phase angle - δ	Lowest value of $\left[\Omega^2 \frac{(Rl_1)^4}{EI} q\right]^{\frac{1}{4}}$ from present method	Corresponding lowest value of $\left[\Omega^2 \frac{(Rl_1)^4}{EI} q\right]^{\frac{1}{4}}$ as obtained by Miles (13)
1	π	3.1423	3.1428
2	2π	6.2811	6.2856
3	$\pi/3$	6.90	6.7086

The frequency parameter for a infinite continuous beam when the phase angle is π or 2π is coming as π or 2π respectively. This is just expected because when the inter-panel phase angle is π or its integer multiple then the modes of vibration are identical with those of a single span of simply - supported beam. These situations have been deal with in detail by Ayre and Jacobson (17).

The lowest value of λ for $\sigma = \pi/3$ as obtained by Miles (13) is 4.297 and the next higher value is 6.7086. It is the later value of λ which is corresponding to the value of λ for $\sigma = \pi/3$ obtained by the present method. While analysing the stiffened rings only inextensional modes of vibrations have been considered here and when the formulation is extended to that of a fully - infinite continuous beam we essentially miss those modes of vibration which are extensional modes for a beam.

Natural Frequencies of Rings:

Natural frequencies have been obtained for circular stiffened rings having 10 and 20 radial supports and for phase angles of π, 2π and $\pi/3$ and the results are given in Table II. As discussed in Chapter III the situations when σ is π or its integer multiple is essentially an uncoupled mode and all the spans vibrate independently as a single span curved beam. A.E.H. Love (5) has obtained the frequency equation for a free - ring as:

$$w^2 = \frac{E\ I\ g}{A\ R^4}\ \frac{n^2(n^2 - 1)^2}{n^2 + 1} \tag{4.3}$$

where n is the number of full waves. Rao and Sundararajan (8) have given the natural frequencies of vibration of a free - ring as calculated from the above equation. The situation of free vibration of a free - ring with n full waves as discussed by Love (5) and the free vibration of a stiffened ring with 2n radial supports having a phase angle of π are just identical. Similarly the situation of ffee vibration of a free - ring with n full waves and the free vibration of a stiffened ring with n radial supports having a phase angle of 2π are same. An inspection of Table II confirms the correctness of the present formulation.

TABLE 2

FUNDAMENTAL FREQUENCIES OF VIBRATION FOR A STIFFENED CIRCULAR RING WITH 10 RADIAL SUPPORTS

Serial Number	Values of phase angle - σ	Lowest value of $\Omega^2\left[\frac{R^4}{E\,I}q\right]^{\frac{1}{2}}$	Corresponding value of $\Omega^2\left[\frac{R^4}{E\,I}q\right]^{\frac{1}{2}}$ from the classical theory.
1	π	23.85	23.5
2	2π	98.75	98.50
3	$\pi/3$	118.4	Data not available

TABLE 3

FUNDAMENTAL FREQUENCIES OF VIBRATION FOR A STIFFENED CIRCULAR RING WITH 20 RADIAL SUPPORTS

Serial Number	Values of phase angle - σ	Lowest value of $\Omega^2\left[\frac{R^4}{E\,I}q\right]^{\frac{1}{2}}$ from present method	Corresponding value of $\Omega^2\left[\frac{R^4}{E\,I}q\right]^{\frac{1}{2}}$ from the classical theory.
1	π	98.85	98.50
2	2π	399.0	398.5
3	$\pi/3$	495.0	Data not available

Table 2 and Table 3 give the lowest natural frequencies for a ring having 10 and 20 radial supports respectively and with phase angles of π, 2π and $\pi/3$. The results with π and 2π phase angle compare very well with those obtained from the classical theory.

The present work can be further extended to analyse the extensional modes of vibration for the stiffened - ring.

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APPENDIX I

Generalisation of Reduction Procedure to Infinite Degrees of Freedom:

Stearman (14) has derived the equations for the flutter of cylindrical shell panels. We present the same analysis modified for the ring problem.

The equation of motion of a single span of a stiffened ring under free vibration conditions is given in the form

$$w(l) - \Omega^2 \int k(l,m)\, w(m)\, dm = 0 \qquad (A1 - 1)$$

where $w(l)\, e^{j\omega t}$ is the displacement at the point l and $k(l,m)$ is the influence function (Ref. Equation - 2.1).

For N such identical spans cyclically arranged at equal spacing over a circle takes the form of equation (2.4) where $\underline{w}$ is a column matrix and $\underline{k}$ is a circulant square matrix.

To shorten the writing we introduce the so called composition product of $\underline{k}$ and $\underline{w}$ defined as follows:

$$\underline{k} * \underline{w} \equiv \int \underline{k}(l,m)\, \underline{w}(m)\, dm \qquad (A1 - 2)$$

We again introduce the unitary element I such that

$$\underline{I} * \underline{f} = \underline{f} * \underline{I} = \underline{f} \qquad (A1 - 3)$$

where $\underline{f}$ is any function compatible with the composition product definition. Now (A1 - 1) can be written in the matrix

equation form as

$$\underline{H} \cdot \underline{y} = 0 \qquad \text{(A1-4)}$$

where

$$\underline{H} = \underline{I} - \Omega^2 \underline{k} \qquad \text{(A1-5)}$$

Since $\underline{k}$ is a circulant matrix, it is evident that $\underline{H}$ is also a circulant matrix.

Now let us consider the alternant matrix $\underline{P}$ such that

$$\underline{P} = \begin{bmatrix} 1 & \omega_0^{-1} & \omega_0^{-2} \cdots\cdots & \omega_0^{-n+1} \\ 1 & \omega_1^{-1} & \omega_1^{-2} \cdots\cdots & \omega_1^{-n+1} \\ \cdots & \cdots & \cdots & \\ 1 & \omega_{n-1}^{-1} & \omega_{n-1}^{-2} \cdots\cdots & \omega_{n-1}^{-n+1} \end{bmatrix} \qquad \text{(A1-6)}$$

where ω_ρ be the $\rho^{th.}$ of the n $n^{th.}$ roots of unity i.e.

$$\omega_\rho = e^{2\pi j(\rho/n)}; \qquad (\rho = 0, 1, 2, \ldots, n-1) \qquad \text{(A1-7)}$$

$$j = (-1)^{\frac{1}{2}}$$

so that $\omega_\rho^n = 1$

The reciprocal of $\underline{P}$ is given as

$$\underline{P}^{-1} = \frac{1}{n}\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \omega_0 & \omega_1 & \cdots & \omega_{n-1} \\ \omega_0^2 & \omega_1^2 & \cdots & \omega_{n-1}^2 \\ \cdots & \cdots & & \\ \omega_0^{n-1} & \omega_1^{n-1} & \cdots & \omega_{n-1}^{n-1} \end{bmatrix} \qquad \text{(A1- 8)}$$

Now it can be verified by direct multiplication that

$$\underline{P}\,\underline{A}\,\underline{P}^{-1} = \underline{B} \qquad \text{(A1- 9)}$$

where $\underline{A}$ is any circulant matrix then $\underline{B}$ is a diagonal matrix. And since $\underline{A}$ and $\underline{B}$ are related by a collineatory transformation, the eigen values of $\underline{A}$ and $\underline{B}$ are identical.

The reduction of a circulant determinant $\underline{A}$ into a constant multiple of $\underline{B}$ was first announced without proof by W-Spottiswoode in 1853. The proof by means of the alternant determinant $\underline{P}$ is given by L. Cremona (1856) who attributes the proof to Brioschi.

It can be further shown by actual multiplication that

$$\underline{P}\,\underline{I} * (\underline{H} * \underline{v}) = (\underline{P}\,\underline{I} * \underline{H}) * \underline{v} \qquad \text{(A1- 10)}$$

and also

$$\underline{P}\,\underline{I} * \underline{H} = \underline{B} * \underline{P}\,\underline{I} \qquad \text{(A1- 11)}$$

where $\underline{B}(1,m)$ is a diagonal matrix of which the $j^{th.}$ element on the principal diagonal is

$$B_j(l,m) = H_1(l,m) + \omega_j^{-1} H_n(l,m) + \omega_j^{-2} H_{n-1}(l,m) \qquad \text{(A1-12)}$$

$$+ \cdot \cdot \cdot \cdot \cdot \cdot + \omega_j^{-n+1} H_2(l,m)$$

$$(j = 0,1,2,\ldots\ldots,n-1)$$

or

$$B(\sigma_i) = H_1 + H_2 e^{j\sigma_i} + H_n e^{-j\sigma_i} + H_3 e^{j2\sigma_i} + H_{n-1} e^{-j2\sigma_i} \ldots$$

Now by combining (A - 9) to (A - 11) it can be shown that

$$\underline{P}\,\underline{I} * (\underline{H} * \underline{w}) = \underline{B} * \underline{p} = 0 \qquad \text{(A1-13)}$$

where $\underline{p} = \underline{P} * \underline{w}$

It can be shown that the eigen value problems $\underline{H} * \underline{w} = 0$ and $\underline{B} * \underline{p} = 0$ are completely equivalent.

Thus the $n^{th.}$ order coupled eigen value problem is reduced into solving n uncoupled problems of equivalent single panels.

APPENDIX II

Spanwise Eigen Functions for the Vibration Analysis of Stiffened Rings:

The differential equation of motion for a continuous beam of equal spans is given as

$$\frac{d^4 w^{(i)}}{(dl^{(i)})^4} - \lambda^4 l^{(i)} = 0 \; ; \; i = 1, 2, \ldots\ldots\ldots, N \qquad (A2 - 1)$$

with the time factor $e^{j\omega t}$ removed.

For a fully - infinite beam of equal spans with the following boundary conditions:

$$w^{(i)}(0) = w^{(i)}(1) = 0 \qquad (A2 - 2)$$

$$\frac{dw^{(i)}}{dl^{(i)}}(0) = \frac{dw^{(i)}}{dl^{(i)}}(1) \qquad (A2 - 3)$$

and $$\frac{d^2 w^{(i)}}{(dl^{(i)})^2}(0) = \frac{d^2 w^{(i)}}{(dl^{(i)})^2}(1) \qquad (A2 - 4)$$

the general solution to equation (A2 - 1) which satisfies all the boundary conditions (A2 - 2 to A2 - 4) has been derived by Miles (13) to be

$$w^{(i)}(l) = f_s(l)\, e^{jn\sigma} + f_s(1 - l)\, e^{j(n-1)\sigma} \qquad (A2 - 5)$$

where $f_s(l) = \text{Sinh}\ \delta(s)\ \text{Sin}\ \delta(s)l - \text{Sin}\ \delta(s)\ \text{Sinh}\ \delta(s)l$ (A2 - 6)

and

$$\mathrm{Cos}\,\sigma = \frac{\mathrm{Sinh}\,\delta(s)\,\mathrm{Cos}\,\delta(s) - \mathrm{Sin}\,\delta(s)\,\mathrm{Cosh}\,\delta(s)}{\mathrm{Sinh}\,\delta(s) - \mathrm{Sin}\,\delta(s)} \qquad (A2 - 7)$$

If σ is an integer multiple of π then equation (A2 - 5) becomes indeterminate. To avoid indeterminacy, it is found convenient to treat separately those modes that are identical to single span beam viz.

$$w^{(i)}(l) = \mathrm{Sin}\,\delta(s)l \text{ where } \mathrm{Sin}\,\delta(s) = 0 \qquad (A2 - 8)$$

This is also a solution to (A2 - 1) and satisfied all the boundary conditions (A2 - 2 to A2 - 4).

Restoring the time factor $e^{j\omega t}$, the solution (A2 - 5) represents a disturbance travelling in the negative or positive l direction as σ is positive or negative respectively.

The wave length of the disturbance is given by $2\pi L/\sigma$ where L is the span length and it is integrally related to L only when σ is an integer multiple of π. The phase - velocity is given by

$$V = \frac{\omega \times L}{\sigma} \qquad (A2 - 9)$$

In the case of stiffened rings on large number of supports the infiniteness of i replaces the usual cyclic boundary condition of closure required on a full ring with small number of supports. Hence the above system of equations can be used without any loss of generality as the spanwise Eigen function for the stiffened rings,with,the suitable curvilinear coordinates.

The spectrum of frequencies for the above system has a filter like character, being made up of discrete bands with a lower cut off. These frequencies pass bands lie approximately in the interval

$$s\pi \leqslant \delta(s) \leqslant (s+\tfrac{1}{2})\pi + 2\,(-1)^{s+1}\,e^{-(s+\frac{1}{2})\pi} + O\,(e^{-(2s+1)}\quad) \qquad (A2-10)$$

in equation (A2 - 5) is the phase angle between two adjecent spans and has been plotted in Fig. 5 by equation (A2 - 7).